Spontaneous emission of a continuum sine-Gordon kink in the presence of a spatially periodic potential

C. R. Willis
Department of Physics, Boston University, Boston, Massachussett 02215
(Received 19 September 1996)

We find the spontaneous emission of radiation of a sine-Gordon kink perturbed by a spatially periodic potential by solving the lowest order coupled collective variable and phonon equations. We show that the radiation rate is finite, and that the qualitative properties of the kink and the radiation are a special case of the phenomena that occur at the lower band edge in the discrete sine-Gordon equation. Thus we solve the radiation threshold problem for a sine-Gordon kink in a periodic potential for weak coupling. In the Doppler limit we calculate the rate of decay of the kink velocity due to phonon emission, and show that the rate is a generalized power law. [S1063-651X(97)04105-6]

PACS number(s): 03.40.Kf

I. INTRODUCTION

There are many nonlinear problems which can be described by the addition of potentials or inhomogeneity to a nonlinear Klein-Gordon equation which possess kink solutions. The additions will usually destroy the integrability if it exists and will alter the nature of the kink solutions. In the case of the discrete sine-Gordon (SG) the effect of the discreteness is to alter the shape of the kink, cause the kink to radiate phonons and to cause the kink to be trapped in a well of the Peierls-Nabarro (PN) potential. A very useful aid to understanding the radiation and trapping phenomena for an arbitrary potential or inhomogeneity is to express the original problem in terms of one or more collective variables and the phonon degrees of freedom. One finds that the radiation from the trapped case is very different than that from the untrapped case. The frequency in the trapped case is determined by the curvature of the PN well and by the Doppler frequency in the untrapped case. In the Doppler limit where the kinetic energy $\gg$ potential energy the kink radiates phonons and damps due to spontaneous emission. The Doppler radiation problem is analogous to the problem of calculating the radiation relaxation time of an accelerating electron due to radiation of electromagnetic waves. In fact, in this paper we use the same historical method used to calculate the radiating electron relaxation time. The only significant qualitative difference of the electron and nonlinear Klein-Gordon cases is that the kink case has a phonon cutoff at the finite lower band edge, so that the Doppler radiation ceases when the velocity of the kink decays to the value of the velocity which causes the Doppler frequency to equal the frequency of the lower band edge. We will treat the continuum sine-Gordon kink so there is a single length scale in the unperturbed problem, i.e., the length of the kink. The perturbation or inhomogeneity may have many length scales. For clarity of presentation we take a spatially period perturbation of a single length scale k^{-1}.

Consequently, here we solve for the spontaneous emission of a continuum SG kink and calculate the radiation damping caused by a spatially periodic potential $\epsilon\cos kx$ for small ϵ using the lowest order in ϵ coupled phonon and collective variable (CV) equations of motion. Consequently, we solve the radiation threshold problem of a SG kink in a periodic potential. We show the radiation caused by $\epsilon\cos kx$ in the continuum is essentially a special case of the radiation phenomena that occurs at the lower band due to discreteness that occurs in the discrete SG equation. The SG equation in the presence of $\epsilon\cos kx$ is

$$\ddot{\phi}-\phi''=\frac{dV}{d\phi}=-(\pi/l_0)^2(1+\epsilon\cos kx)\sin\phi, \tag{1}$$

where the dot (prime) represents the derivative with respect to time (space). We introduce collective variables $X(t)$ representing the center of mass of the kink, and $\Gamma(t)$ representing the slope of the kink with the ansatz

$$\phi(x,t)=\sigma[\Gamma(t)(x-X(t))]+\chi[\Gamma(t)(x-X(t)),t] \tag{2}$$

where $\sigma(y)\equiv\tan e^{y}$. In Ref. [1] the rigorous derivation of the equations of motion for $\chi(t)$, $X(t)$, and $\Gamma(t)$ are given for the case $\epsilon=0$. The derivation for $\epsilon\neq0$ consists simply in replacing $(\pi/l_0)^2[1-\cos\phi]$ of Ref. [1] by $(\pi/l_0)^2(1+\epsilon\cos kx)[1-\cos\phi]$.

For relativistic velocities it is necessary to have the collective variable $\Gamma(t)$. In the present paper, for convenience, we consider the nonrelativistic or slightly relativistic case by taking $\Gamma(t)=(\pi/l_0)\gamma(t)$ where $\gamma\equiv(1-X^2)^{-1/2}$, i.e., we treat Γ as a kinematic term instead of an independent dynamical variable. Since we are considering spontaneous emission caused by $\epsilon\cos kx$ with ϵ small, we can linearize Eq. (1) for χ with respect to ϵ, and obtain

$$\ddot{\chi}-\chi''+(\pi/l_0)^2\cos\sigma\chi\equiv\ddot{\chi}+L\chi=-\epsilon\sigma''(\pi/l_0)^2\cos kx, \tag{3}$$

where prime on σ' represents derivative with respect to the argument of σ, and where we used $\sigma''=\sin\sigma$. When we substitute Eq. (2) for ϕ in Eq. (1), multiply by σ', and integrate over x, we obtain the equation of motion for the CV, $X(t)$:

$$M_x\ddot{X}=\left\langle \sigma' \left| \frac{dV}{d\sigma}\right.\right\rangle=\epsilon(\pi/l_0)^2\int \sigma'(\xi)\sigma''(\xi) \times \cos k(\xi/\Gamma+X)\,d\xi \tag{4}$$

where

$$M_x\equiv\Gamma\langle\sigma'|\sigma'\rangle=8\Gamma, \quad \langle A|B\rangle\equiv\int A(\xi)B(\xi)d\xi.$$

There are no terms proportional to χ on the right-hand side of Eq. (4) to first order in ϵ, because the kink does not interact further with the phonons the kink radiates as described by Eq. (3). Since we are considering the case where all the phonons come from spontaneous emission induced by $\epsilon\cos kx$, so that $\chi\sim\epsilon$, we can also neglect the dynamical dressing of M_x. The two coupled equations (3) and (4) for $\ddot{X}$ and $\ddot{\chi}$ constitute a complete closed collective variable description to order ϵ of Eq. (1) for ϕ with σ given by Eq. (2). There are two constraints [1] that have to be satisfied which are satisfied in the calculation of the radiation below. Equations (3) and (4) conserve the energy of the kink and phonons to order ϵ which leads to the result that there are no divergences in radiation rates in the linear in ϵ theory. In particular, the spurious divergence that appears in the calculation of the phonon radiation rate using just Eq. (3) instead of the coupled equations (3) and (4) arises because the complete linear theory in ϵ requires the energy gain of the phonons to be compensated by energy loss of the kink which cause $\dot{X}$ to be a decaying function of time instead of being a constant, thus invalidating the initial assumption that $\dot{X}$ is independent of time [2].

II. ANALYSIS OF EQUATION OF MOTION FOR $\dot{X}(t)$

We first analyze the nonlinear motion of the kink in the cos *kx* potential neglecting radiation. Evaluating the integrals in Eq. (4), we obtain $\ddot{X}=\omega^2\sin kX$, where

$$\omega^2=(\epsilon/\pi\Gamma)(\pi/l_0)^2\eta^2(\sinh\eta)^{-1}, \tag{5}$$

where l_0 is the size of the kink, λ is the wavelength of the potential, and $\eta\equiv(kl_0/2\gamma)$.

The point $X=0$ is unstable, so we define $2\pi z=kX+\pi$, and Eq. (5) becomes

$$\ddot{z}+(\Omega^2/2\pi)\sin 2\pi z=0,$$

where

$$\Omega^2\equiv k\omega^2=(2\epsilon/l_0^2)\eta^3(\sinh\eta)^{-1}, \tag{6}$$

which is the pendulum equation. The maximum value of the frequency Ω is $\Omega_m\equiv0.74(\epsilon)^{1/2}$, and occurs for $\lambda=l_0$, i.e., when the wavelength of the potential is equal to the size of the kink. We can gain insight into the cos*kx* problem when we observe that Eq. (6) is a special case of the equation of motion for the bare center of mass CV in a discrete SG [3]

$$\ddot{X}+\sum_{n=1}^{\infty}\Omega_n^2\sin 2\pi nX=0 \tag{7}$$

corresponding to the single term $n=1$. The qualitative radiative phenomena that occur at the lower band edge in the discrete SG occur in the continuum SG in the cos*kx* potential. When a harmonic of Ω, $n\Omega$, is greater than the lower band-edge phonon, π/l_0 enters the phonon band for the first time, and there is a burst of phonon radiation [3]. There is a decrease of radiation (called a knee) [4], when the Doppler frequency crosses the lower phonon band edge, so that Doppler radiation is no longer possible, and all that remains is the much weaker anharmonic radiation. The energy of the nonlinear z oscillator is

$$E_z=E_z^0+\tfrac{1}{2}M_z\dot{z}^2+M_z\left(\frac{\Omega}{2\pi}\right)^2[1-\cos 2\pi z], \tag{8}$$

where $M_z\equiv(2\pi/k)^2 8\Gamma$ and the well depth of the potential is $\Delta_z\equiv2M_z(\Omega/2\pi)^2=(8\epsilon\pi/\gamma l_0)\eta(\sinh\eta)^{-1}$. The threshold condition on the velocity for trapping the kink in a potential well is $\dot{z}_T=(\Omega/\pi)$, which corresponds to $\dot{X}_T=\Gamma^{-1}(2\epsilon\eta/\sinh\eta)^{1/2}$.

III. KINK RADIATION

There are three kink radiation regimes. The first is the Doppler regime where $k\dot{X}>(\pi/l_0)$, and where the kinetic energy (KE) is much greater than the potential energy (PE). The completely trapped regime where PE>KE and the intermediate regime where KE and PE are comparable. In the trapped regime where $\dot{X}(t)<\dot{X}_T$, and in the intermediate regime, the kink radiates phonons at harmonics of Ω such that $n\Omega>(\pi/l_0)$. The flux at frequency $n\Omega$ of the radiating trapped kink is proportional to $\epsilon^2(kX_0)^{2n}/(n!)^2$, where $kX_0<1$, the lowest harmonic for small ϵ, is typically $n=4$ or greater. Most of the radiation comes from the harmonic with the smallest n, such that $n\Omega\geqslant1$, where in the remainder of the paper we take units where $\pi=l_0$, so that the band-edge frequency is one in the new units and $\eta\equiv(k\pi/2\gamma)$. The reason for the harmonic nearest the band edge radiating the most is because the phonon density of states is largest at the band edge. In the Doppler regime (with which we are mainly concerned in the paper) we can take the kink to be an almost free particle with constant velocity v, i.e., $\dot{X}=v$, as long as $k\dot{X}\gg1$. The solution of Eq. (3) for χ in the laboratory frame is

$$\chi(x,t)=-2\epsilon\int_0^\infty dl\int_0^t d\tau\; \psi_l^*(x)\omega_l\sin[\omega_l(t-\tau)]d\tau \times\int_{-\infty}^{\infty}\psi_l(\xi-X(t))\sigma''(\xi-X(t))\cos k\xi\; d\xi, \tag{9}$$

where

$$L\psi_l=\omega_l^2\psi_l;\psi_l^*=(2\pi\omega_l)^{-1/2}e^{-ily}(il-\tanh y)$$

and

$$\omega_l^2=(1+l^2).$$

In order to evaluate Eq. (9), we perform a Lorentz transformation from the laboratory frame to the rest frame of the kink, because the phonons are orthogonal to the kink shape mode in the rest frame and thus the constraints on χ are satisfied in the rest frame. The Lorentz transformation is $x' = \gamma(x - vt)$, $t' = \gamma(t - x/v)$, $k' = \gamma(k - \omega/v)$ and $\omega' = \gamma(\omega - kv)$. In the laboratory frame the potential $\cos kx$ has no time dependence, so $\omega = 0$, and $\cos kx$ becomes $\cos \gamma k(x' - vt')$. The ξ integral in Eq. (9) becomes

$$(2\pi\omega_l)^{-1/2}\int d\xi e^{il\xi}(il - \tanh\xi)\sigma''(\xi)\cos\gamma k(\xi - X(t'))$$

$$\approx \pi(\omega_l^2 - \gamma^2 k^2)(2\pi\omega_l^2)^{-1/2}\left\{e^{i\gamma kvt'}\operatorname{sech}\left[\left(\frac{\pi}{2}\right)(\gamma k - l)\right] + (k \leftrightarrow -k)\right\}, \tag{10}$$

where the approximation represents the replacement of $X(t) = \int^t v(\tau)d\tau$ with time-dependent $v(t)$ by the constant v, $X(t) = vt$, which is valid as long as v, which is a damped oscillatory function of time, can be treated as approximately time independent. When we substitute Eq. (10) into Eq. (9), we obtain

$$\chi(x',t') = (2\pi)^{1/2}\epsilon \text{ Re} \int_0^\infty dl\ \psi_l^*(x',t') \times \int_0^{t'} d\bar{t}\sin\omega_l(t' - \bar{t})(\gamma^2k^2 - \omega_l^2) \times \left\{\operatorname{sech}\left(\frac{\pi}{2}\right)(\gamma k - l)e^{i\gamma kvt} + (k \leftarrow -k)\right\}. \tag{11}$$

When we let $t' \to \infty$ in the $\bar{t}$ integration, we obtain the $\delta\pm$ functions where $\delta\pm(\alpha) \equiv \int_0^\infty e^{\pm i\alpha\tau}d\tau = \pi\delta(\alpha) \pm iP(1/\alpha)$, and where P means the principal part and $\alpha \equiv \omega_l + \gamma kv$. The result of performing the time integration is

$$\chi(x',t') = -\frac{\epsilon}{2}\int_0^\infty dl(\gamma^2k^2v^2 - \omega_l^2)\omega_l^{-3}\operatorname{sech}\left(\frac{\pi}{2}\right)(\gamma k - l) \times \left\{\left[\pi l\delta[\omega_l - \gamma kv] + \tanh x' P\left(\frac{1}{\omega_l - \gamma kv}\right)\right] \times \cos(lx' - \omega_l t') + \left(\pi \tanh x'\, \delta[\omega_l - \gamma kv] + lP\left(\frac{1}{\omega_l - \gamma kv}\right)\right)\sin(lx' - \omega_l t') + (k \leftrightarrow -k)\right\}. \tag{12}$$

The term with $+k$ $(-k)$ is the wave going to the right (left). Usually the δ function term is larger than the principal part term. Consequently, we evaluate the δ function terms which require the introduction of the phonon density of states $\rho(\omega_l)d\omega_l = dl$, where $\rho(\omega_l) = (\omega_l/l)$. The argument of the δ function vanishes when ω equals the Doppler frequency γkv, and thus for $\bar{l} = [(\gamma kv)^2 - 1]^{1/2}$. The solution for χ in the kink's rest frame is

$$\chi(x',t') = -\epsilon\pi(2\gamma v^2\bar{l})^{-1}\{\operatorname{sech}[(\pi/2)(\gamma k + \bar{l})] \times [\bar{l}\cos(\bar{l}x' - \gamma kvt') + \sin(\bar{l}x' - \gamma kvt')] + (k \leftrightarrow -k)\}. \tag{13}$$

We calculate the energy flux $\bar{S}$, which is defined as the time average of the product $(\dot{\chi}\chi')_{\text{avg}}$, and obtain the net flux

$$\bar{S} = (\epsilon\pi)^2k^3(8\gamma v\bar{l})^{-1}\{\operatorname{sech}^2[(\pi/2)(\gamma k + \bar{l})] + \operatorname{sech}^2[(\pi/2)(\gamma k - \bar{l})]\} \tag{14}$$

evaluated at a distance greater than the size of the kink, where $\tanh x' = 1$, and where the first (second) term in the brackets is the flux to the right (left). In the limit $\bar{l} \to 0$ the right and left fluxes become equal in magnitude. More importantly, as $\bar{l} \to 0$, the band edge, the flux diverges, i.e., the rate of increase of phonon energy becomes infinite, which is impossible for the solution of Eqs. (3) and (4) because the energy of kink plus phonons is conserved and we start initially with a finite energy kink. The divergence results because the condition for the Doppler limit, i.e., the velocity of the particle can be taken as effectively constant is no longer true when $\bar{l} \to 0$. Since the energy of kink plus radiation is conserved, the kink loses energy at the negative of the rate of increase of the phonon energy. Consequently, $\dot{X}(t)$ becomes time dependent, and decreases appreciably as $\bar{l} \to 0$. The cause of the problem is the phonon density of states, and the δ function which results from taking the velocity of the kink to be constant. There is no divergence when we use both Eqs. (5) and (9), which entails a time-dependent velocity, because there is no longer a δ function in frequency, and the resultant frequency integral is over a smooth function of frequency which is integrable at the $\bar{l} \to 0$ limit.

IV. TIME DEPENDENCE OF $\dot{X}$ DUE TORADIATION DAMPING

We can use the conservation of total energy to determine the time dependence of the kink velocity in the limit where KE$\gg$PE, because we can neglect the PE of the kink, and use the fact that the time rate of change of the kink's KE is the negative of the total flux of phonons which is proportional to ϵ^2; thus the velocity is slowly varying compared to the Doppler frequency γkv, which is $\gg 1$:

$$\frac{d}{dt}\left(\frac{1}{2}M\dot{X}^2\right) = -(\epsilon\pi)^2k^3(8\bar{l}V)^{-1}\{\operatorname{sech}^2(\pi/2)(\gamma k + \bar{l}) + \operatorname{sech}^2(\pi/2)(\gamma k - \bar{l})\}. \tag{15}$$

Define $\mu \equiv \gamma k\dot{X}$, use $\bar{l} = (\mu^2 - 1)^{1/2}$, and substitute in Eq. (15), which becomes $\mu^3(\mu^2 - 1)^{1/2}d\mu = -\tau^{-1}dt$, where $\tau \equiv 32[(\epsilon\gamma\pi \operatorname{sech}(\pi/2)\gamma k)^2k^6]^{-1}$, where we set l equal to zero in the argument of the sech because the dominant contribution comes in the neighborhood of the band edge. The condition for the validity of the first Born approximation is $\tau \gg (\gamma kv)^{-1}$, which is satisfied for sufficiently small ϵ, and is of the same form as the usual Born approximation for spontaneous emission $\omega\tau \gg 1$. The solution of Eq. (15) is

$$\frac{4(t-t_f)}{\tau}=-\mu(\mu^2-1)^{1/2}(\mu^2-\tfrac{1}{2})-2^{-1}\ln[\mu+(\mu^2-1)^{1/2}] \quad (16)$$

where t_f is the time it takes the kink to lose energy, such that $\mu=\gamma k\dot{X}(t_f)=1$, at which point the kink stops radiating by the Doppler mechanism. The time t_f is a function of only $\mu_0=\gamma k\dot{X}(0)$, where $\dot{X}(0)$ is the initial velocity, and t_f is determined by setting $t=0$ in Eq. (16) and using the definition of τ. For typical values of $\dot{X}(0)$, $t_f\sim 0(\tau)$. The minimum value of τ occurs for $\gamma k\sim 3$ which corresponds to $\lambda\sim\gamma l_0$, i.e., to a λ about the size of a kink. When the damped time dependence of Eq. (16) for $\dot{X}(t)$ is substituted into $X(t)=\int^t\dot{X}(t')dt'$, and $X(t)$ is inserted into Eq. (9), the radiation rate remains finite.

The approach to spontaneous emission in this section is essentially the same as the elementary historical approach to the problem of radiation damping of an accelerated electron, where the damping is calculated by using the conservation of energy of the particle plus radiation; e.g., see Ref. [5]. The acceleration of the kink in this case is caused by the $\sin 2\pi z$ term in Eq. (6).

V. DISCUSSION AND CONCLUSION

The same type of behavior is observed in the decay rate of the velocity of the kink in the discrete radiation problem [3] where a complicated power-law type of decay of $\dot{X}(t)$ is also observed. In both the present $\cos kx$ potential problem and the discrete kink problem when $\gamma k\dot{X}$ is less than 1, there is a sharp drop in the rate of radiation called a ''knee'' [4] because there is no longer any Doppler radiation, and all that remains is weaker radiation which is always present due to the nonlinear harmonics $n\Omega$ which are resonant with the phonon band. However, in the discrete SG there is more than one knee, i.e., there is a knee for each n in the sum in Eq. (7) while there is only one knee in the continuum SG with a spatially periodic potential.

Essentially the same flux $\bar{S}$, Eq. (14), was obtained in Ref. [6] by finding a different Green's function for Eq. (1) linearized about the kink solution with constant velocity. The authors of Ref. [6] treated the constant velocity case only, and their solution was not valid in the neighborhood of the band edge. If external damping and driving are added to Eqs. (3) and (7), Refs. [7] and [8], the fundamental problems that exist in the undamped and undriven case do not occur because (1) the damping changes the δ function that occurs in Eq. (12) into a Lorentzian of width α, where α is the external damping constant which leads to a finite damping rate of the kink proportional to α^{-1}; and (2) in the steady state the velocity of the kink becomes a constant independent of time.

In conclusion, the first order in ϵ coupled equations of motion for the kink center of mass, $X(t)$, and phonons, $\chi(t)$, solves the radiation threshold problem that was raised in earlier papers and thoroughly discussed in Ref. [9]. Our results for $\gamma kv<1$, when there is no Doppler radiation, indicate that there is a weak anharmonic radiation for the nth harmonic where $n\Omega>1$ due to the nonlinearity of the $\sin kX$ motion. When $\gamma kv>1$ and KE≫PE, there is strong Doppler radiation because the Doppler frequencies are resonant with frequencies in the phonon band in lowest order, i.e., the first. If $\gamma kv>1$ and KE>PE, there are both Doppler and anharmonic radiation present. In the Doppler limit when γkv becomes equal to 1, the Doppler radiation ceases, and there a sharp drop in the radiation rate called a knee. In Ref. [9], where the same model was used, the knee in the radiation rate was missed because all the frequencies in the simulations corresponded to Doppler frequencies in the gap and not in the phonon band. Consequently, the only radiation was the anharmonic radiations of the kinks in the intermediate region. In the Doppler limit, we calculated the generalized power law radiation damping of the kink. The qualitative behavior of the continuum $\cos kx$ perturbation of the SG is a special case of the phenomena that occur at the lower band edge in the discrete SG problem. For the continuum SG there is no upper band edge as there is in the discrete SG problem. There are phenomena such as a kink scattering off a well, changing direction, and returning a few wells before trapping, caused by the kink absorbing radiation it had previously emitted, which required higher order equations in ϵ to describe. Such phenomena were observed in the discrete SG [3], where the simulations were to all orders in ϵ. The present method applies to any nonlinear Klein-Gordon equation, whose eigenfunctions and eigenvalue obtained by linearizing about a kink solution, are known. Finally, the result holds as long as the KE≫PE, even if ϵ is finite.

[1] R. Boesch and C. R. Willis, Phys. Rev. B **42**, 2290 (1990).
[2] R. Boesch, P. Stancioff, and C. R. Willis, Phys. Rev. B **38**, 6713 (1988).
[3] R. Boesch, C. R. Willis, and M. El-Batanouny, Phys. Rev. B **40**, 2284 (1989).
[4] M. Peyrard and M. D. Kruskal, Physica D **14**, 99 (1984).
[5] R. H. Good and T. J. Nelson, *Classical Theory of Electric and Magnetic Fields* (Academic, New York, 1971).
[6] G. S. Mkrtchyan and V. V. Schmidt, Solid State Commun. **30**, 791 (1979).
[7] B. A. Malomed, Phys. Lett. A **144**, 351 (1990).
[8] B. A. Malomed and M. I. Tribelsky, Phys. Rev. B **41**, 11 271 (1990).
[9] A. Sánchez, A. R. Bishop, and W. Dominguez-Adame, Phys. Rev. E **49**, 4603 (1994).